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GENERALIZED HIT-MISS OPERATIONS

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Abstract

The morphological operations of a hit-miss transform, opening and closing are generalized in a number of ways. The new operations have been applied to a variety of binary image analysis problems that involve pattern detection and reconstruction. Generalized openings are developed by replacing erosions with hit-miss transforms. These new openings are shown to be anti-extensive, idempotent, and center-independent. Similarly generalized closings are developed and related to openings by duality. Additionally, the hit-miss transform is further generalized by replacing the erosions with blur and rank order transforms, in order to improve the robustness of pattern matches. The set of invariance properties of these new transforms can be widened by forming generalized openings from them.

1 Introduction

The morphological opening and closing operations satisfy several elegant and useful properties, such as center-independence, idempotence, and anti-extensivity or extensivity, in addition to the general morphological properties of translation invariance and increasing. However, because they are restricted to only "hits" and "don't-cares" in the *structuring element* (abbreviated as SE), they are severely limited in generality. The *hit-miss transform (HMT)*, on the other hand, is an extremely general pattern-matching operation. It is a generalization of the erosion to SEs that specify "misses" as well as "hits", but it lacks even the increasing property of the erosion.

The opening operation is an erosion followed by a dilation with the same SE. It is center-independent because the image translations for the erosion and dilation, specified by the SE, are in opposite directions (see definitions in (1) and (2)). The opening is idempotent because each pixel remaining after the erosion represents a pattern match of the SE element to the original image, and is subsequently dilated by the SE to reproduce exactly those pixels in the original image that were responsible for the initial pattern match. When the opening is repeated, the erosion re-converts each dilated set of pixels to the same set that was obtained after the first erosion. The opening is also anti-extensive since the opened image is always contained within the original image.

The initial motivation for this work was to construct generalizations of the opening and closing operations that include SEs with both hits and misses, and that share most of the special properties of the standard opening and closing operations. From the foregoing, one might guess that an

HMT followed by a dilation by only the hits in the SE would be one such generalization for the opening. This generalized opening would reproduce all the hits in the original image for which the HMT gives an exact pattern match. Thus, it would extract shape features, in their entirety, from the image, and the result would be a fixed point of simple operations related to the specified shapes. Use of such generalized openings, with SEs composed of both hits and misses, implicitly broadens our view of the patterns that are being matched. When SEs are composed of hits only, they are naturally viewed in terms of *shapes* to be matched in the image. But when misses are introduced into the SE, we can alternatively consider that the patterns being matched are short-range *textures*. The generalized opening then extracts textural components, again in their entirety. The dividing line between shape and textural properties is not well defined, but short-range texture can be intuitively understood as the local geometric relations between hits and misses in the image. This is exactly what is specified by the SE.

In the standard opening and closing operations, the foreground and background pixels are implicitly treated differently by using SEs that specify only foreground pixels. However, when generalizing from the HMT, where both foreground and background pixels are specified, one may also wish to consider the case where the subsequent dilation involves the background pixels.

Corresponding to any generalized opening operation, there is a dual generalized closing that must satisfy the same general set of properties as the generalized opening, and in fact is equivalent to a generalized opening on the background with an appropriately transformed SE.

Similar generalizations can also be made by using *rank order* operations. A rank order operation on a binary image is equivalent to a thresholded convolution by a binary SE. When the SE consists only of hits, as in the usual definition, the rank order operation is shift-invariant and increasing, and the morphological erosion and dilation are special cases. Rank order operations are useful for pattern matching because they have greater immunity than erosion to shape distortion, but they are more complicated computationally. When a rank order operation is followed by a dilation by the SE, a type of opening results where shapes, specified by the SE that are only partially matched by the rank order, are reproduced in their entirety on the resulting image. In analogy with the generalized openings derived from the HMT, the HMT can first be generalized to a rank order operation that uses a SE with both hits and misses. This rank order operation can subsequently be generalized to an opening by dilating the result of the rank order by only the hits in the SE. As with the simpler morphological operations, the motivation for using such operation sequences is the expanded set of invariance properties.

In the following sections, we first define and then derive properties of the generalized openings and generalized closings. There are two types of each, that we call *foreground* and *background* openings and closings, and that are simply related to each other. The foreground operations are idempotent, whereas the background operations are only fixed points of opening by the hits. We then discuss two generalizations of the HMT that improve the robustness of the pattern match. The first is a computationally efficient method we call a *blur* match, that gives immunity to noise near shape boundaries. A more flexible method is the thresholded convolution (or rank order filter), that gives an optimum decision for detecting shapes corrupted by salt and pepper noise. Finally, the hit-miss rank order operation is extended to a generalized rank order opening that

allows reconstruction of partially occluded shapes and textures.

2 Generalized Opening

Let the planar set X represent a binary image and let the compact set A be a structuring element. The *erosion* $\ominus$ and *dilation* $\oplus$ of X by A are defined as

$$X \ominus A = \{z : A + z \subseteq X\} = \bigcap_{z \in A} X - z \tag{1}$$

$$X \oplus A = \{z : \check{A} + z \subseteq X\} = \bigcup_{z \in A} X + z \tag{2}$$

where $\check{A} = \{-a : a \in A\}$ is the *reflection* of A with respect to the origin and $X \pm z = \{x \pm z : x \in X\}$ is the *translation* of X along the pixel vector $\pm z$.

The HMT of X by a disjoint pair (A, B) of SEs is defined in [11] as the set transformation

$$X \otimes (A, B) = (X \ominus A) \cap (X^c \ominus B) \tag{3}$$

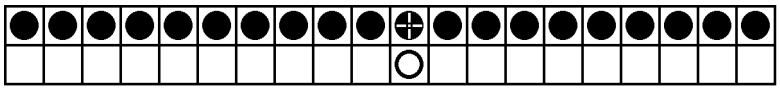
where A is the "hit" SE and B is the "miss' SE. By "hits" we will mean in this paper intersection with the foreground of X, whereas the "misses" will refer to intersection with the background of X, i.e., the set complement X^c. Thus the HMT is the intersection of a foreground erosion and a background erosion. For brevity, we will often refer to the disjoint pair (A, B) of SEs as a single SE with both hits and misses. It should be noted that they are both defined with respect to the same center position.

The ordinary opening $X \circ A = (X \ominus A) \oplus A$ of X by A is an erosion followed by a dilation. Replacing the erosion by an HMT leads to what we call a "generalized opening". Specifically, we define the *generalized foreground opening* of X by (A, B) as the set transformation

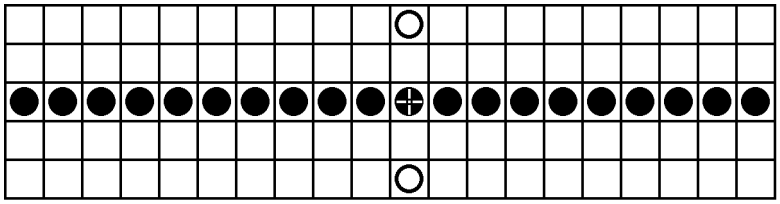
$$\Psi(X; A, B) = [X \otimes (A, B)] \oplus A. \tag{4}$$

Whenever (A, B) are implied, we shall use the simpler notation $\Psi(X)$. Thus the generalized foreground opening is an HMT followed by a dilation with the hit SE A. It is a set consisting of the union of hits for all matches of the HMT.

As an example of the use of the generalized foreground opening, suppose we wish to extract from the image all lower edges of horizontal lines that have a given minimum length of 20 pixels. This can be accomplished with a generalized foreground opening using the SE in Figure 1a. The filled circles are hits and the empty circle is a miss, and the reference point for the SE is indicated by a cross in one of the hits.



(a)



(b)

Figure 1. (a) SE for lower edges of horizontal lines. (b) SE for thin horizontal lines.

When Ψ is applied to the image in Figure 2, the extracted edges are shown in Figure 3a. Because we are using only one miss near the center of the SE, these edges extend into the regions where the horizontal and vertical lines intersect. If such extension is not desired, it can be prevented by placing two misses at the ends of the SE. If we wish to sieve the horizontal lines, finding all horizontal lines of width equal to or less than 3 pixels (for example), the SE in Figure 1b can be used with the generalized foreground opening. The result when applied to the image in Figure 2 is shown in Figure 3b. As expected, the thin lines, in their entirety, have been extracted.

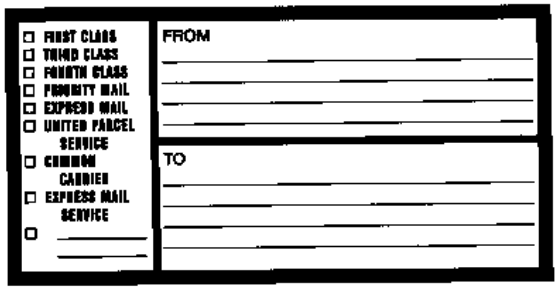
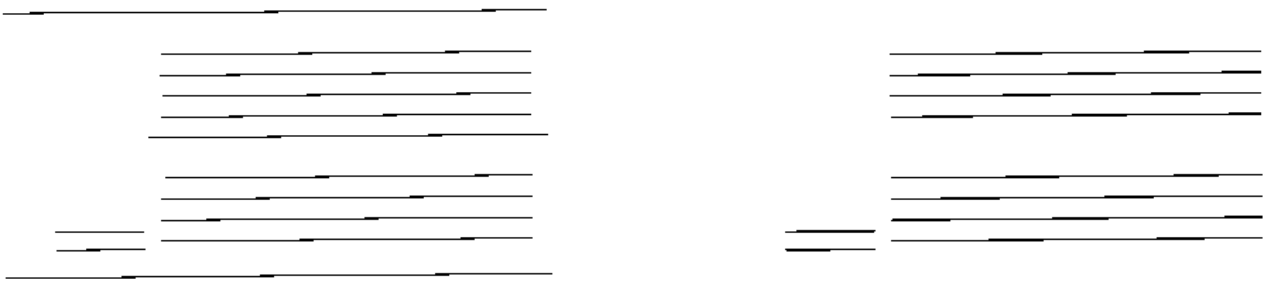


Figure 2. Image composed of small features and horizontal lines. The thin horizontal lines are one pixel wide. Resolution: 135 pixels/inch.



(a) (b)

Figure 3. (a) Gen. foreground opening with SE of Figure 1a.
(b) Gen. foreground opening with SE of Figure 1b.

We now derive several properties of the generalized foreground opening.

PROPERTY 1 . *The generalized foreground opening is anti-extensive; i.e.,*

$$\Psi(X; A, B) \subseteq X \circ A \subseteq X \ . \tag{5}$$

Proof. Since $X \otimes (A, B) \subseteq X \ominus A$ and since dilation is a monotone increasing operator (i.e., $X \subseteq Y \Longrightarrow X \oplus A \subseteq Y \oplus A$), we have

$$\Psi(X) \subseteq (X \ominus A) \oplus A = X \circ A \subseteq X.$$

PROPERTY 2 . *The generalized foreground opening by (A, B) is a fixed point of the opening by A; i.e.,*

$$[\Psi(X; A, B)] \circ A = \Psi(X; A, B) \ . \tag{6}$$

Proof. Since $(S \oplus A) \circ A = S \oplus A$ of any set S, the above property follows by setting $S = X \otimes (A, B)$.

PROPERTY 3 . *The generalized foreground opening is center-independent, i.e., independent of the location of the SE pair; thus, for all vectors z,*

$$\Psi(X; A + z, B + z) = \Psi(X; A, B). \tag{7}$$

Proof. Intuitively, a shift of the center of (A, B) by z causes an equivalent shift in the location of the HMT, and an opposite shift in the location of a dilation. Hence the sequence of HMT and dilation is center-independent. Formally,

$$X \ominus (A + z) = (X \ominus A) - z \;\; ; \;\; X^c \ominus (B + z) = (X^c \ominus B) - z.$$

Hence, $S = X \otimes (A + z, B + z) = [X \otimes (A, B)] - z$. Since $S \oplus (A + z) = (S \oplus A) + z$, the proof of (7) is complete.

By a similar argument as in the previous property, all generalized operations defined in this paper are independent of the location of the SE pair.

PROPERTY 4 . *The generalized foreground opening is idempotent; i.e.,*

$$\Psi[\Psi(X)] = \Psi(X) \; . \tag{8}$$

Proof. First note that, since Ψ is anti-extensive, we have

$$\Psi[\Psi(X)] \subseteq \Psi(X) \tag{9}$$

To prove (8) we need only to show that $\Psi[\Psi(X)] \supseteq \Psi(X)$. Let $Y = X \otimes (A, B)$. Then

$$\Psi(X) \ominus A = (Y \oplus A) \ominus A = Y \bullet A \supseteq Y \tag{10}$$

where $\bullet$ is the ordinary closing. Since $\Psi(X) \subseteq X$ it follows that $[\Psi(X)]^c \supseteq X^c$ and hence $[\Psi(X)]^c \ominus B \supseteq X^c \ominus B$. Then, since $Y \subseteq X^c \ominus B$, we have

$$[\Psi(X)]^c \ominus B \supseteq Y \tag{11}$$

From (10) and (11) it follows that $\Psi(X) \otimes (A, B) \supseteq Y$, which in turn yields

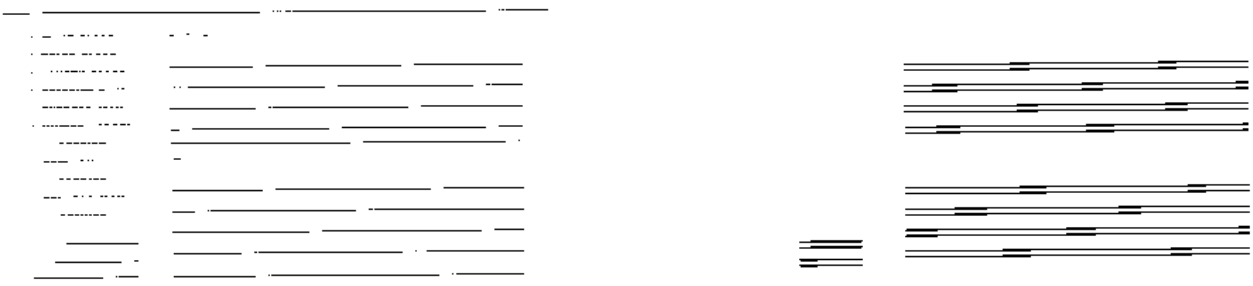
$$\Psi[\Psi(X)] \supseteq Y \oplus A = \Psi(X) \tag{12}$$

Hence, from (9) and (12) it follows that $\Psi[\Psi(X)] = \Psi(X)$, and the proof is complete.

We can also define a *generalized background opening* as the following set transformation $\psi(\cdot)$:

$$\psi(X; A, B) = [X \otimes (A, B)] \oplus B \; . \tag{13}$$

Thus the generalized background opening is an HMT followed by a dilation with the miss SE. As we do for Ψ, whenever (A, B) are implied, we will use the simpler notation $\psi(X)$.



(a) (b)

Figure 4. (a) Gen. background opening with SE of Figure 1a.
(b) Gen. background opening with SE of Figure 1b.

In Figures 4a and 4b, we show the generalized background opening of the image in Figure 2, using the SEs in Figure 1a and 1b, respectively. These sets are contained within the background of Figure 2. The generalized background opening is a set consisting of the union of misses for all matches of the HMT.

PROPERTY 5 . *The generalized background opening of X is a subset of X^c; i.e.,*

$$\psi(X; A, B) \subseteq X^c \circ B \subseteq X^c \ . \tag{14}$$

Proof. Since $X \otimes (A, B) \subseteq X^c \ominus B$ and since dilation is monotone increasing, we have

$$\psi(X) \subseteq (X^c \ominus B) \oplus B = X^c \circ B \subseteq X^c.$$

PROPERTY 6 . *The generalized background opening by (A, B) is a fixed-point of the opening by B; i.e.,*

$$[\psi(X; A, B)] \circ B = \psi(X; A, B) \ . \tag{15}$$

Proof. Same as the proof for (6).

Thus, by performing both operations $\Psi(X)$ and $\psi(X)$, i.e., by dilating the HMT with the hit SE and separately with the miss SE, we obtain two new binary images, which are subsets of the original image foreground and background, respectively. Thus the original image plane is partitioned now into 3 sets of pixels: (1) the pixels of $\Psi(X)$, which are contained in X; (2) the pixels of $\psi(X)$, which are contained in X^c; and (3) the rest, which are the pixels of $[\Psi(X) \cup \psi(X)]^c$.

This information cannot be cast simply as a binary image. We need three different gray levels, one for each of the three classes of pixels. Clearly, the pixel class (3) is the least important and can be treated as the new "background".

The following property reveals that there is a close relationship between Ψ and ψ.

PROPERTY 7 . *The generalized background opening of X by (A, B) is equal to the generalized foreground opening of X^c by (B, A); i.e.,*

$$\psi(X; A, B) = \Psi(X^c; B, A) \ . \tag{16}$$

Proof. From the definitions of Ψ and ψ we have

$$\begin{aligned} \psi(X; A, B) &= [(X \ominus A) \cap (X^c \ominus B)] \oplus B \\ &= [(X^c \ominus B) \cap (X \ominus A)] \oplus B \\ &= \Psi(X^c; B, A). \end{aligned}$$

The center independence of ψ follows from (16) and the center independence of Ψ.

3 Generalized Closing

The duality principle between erosion and dilation, as well as between opening and closing, states that

$$(X^c \ominus A)^c = X \oplus \check{A} \tag{17}$$

$$X \bullet A = (X^c \circ \check{A})^c \tag{18}$$

where $X \bullet A = (X \oplus A) \ominus A$ is the ordinary closing of X by A. Next we introduce a generalized closing based on the duality principle. That is, we define the *generalized foreground closing* of X by (A, B), denoted by $\Phi(X; A, B)$ or simply by $\Phi(X)$ if (A, B) are understood, as follows:

$$\Phi(X; A, B) = [\Psi(X^c; \check{A}, \check{B})]^c \tag{19}$$

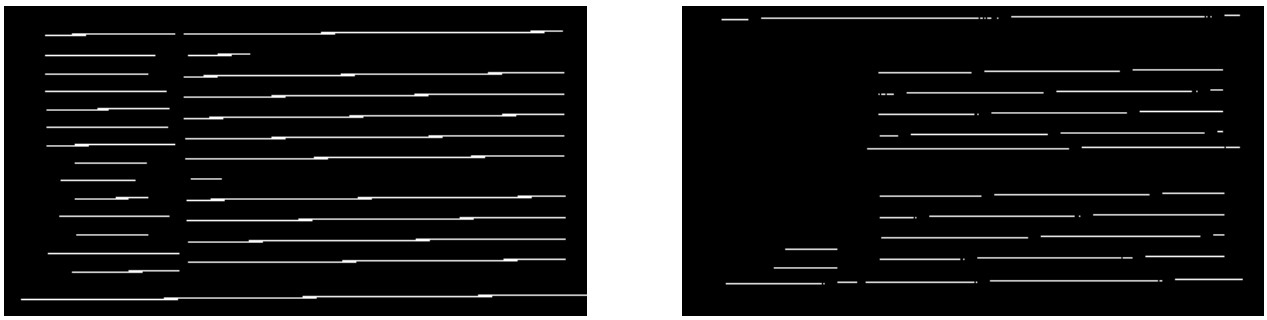
PROPERTY 8 . *The generalized foreground closing is a "dual HMT" followed by an erosion; i.e,*

$$\Phi(X; A, B) = \underbrace{[(X \oplus A) \cup (X^c \oplus B)]}_{\text{dual } \textit{HMT}} \ominus A \tag{20}$$

Proof. From the definitions of Φ and Ψ we have

$$\begin{aligned}
\Phi(X; A, B) &= [\Psi(X^c; \check{A}, \check{B})]^c \\
&= [[(X^c \ominus \check{A}) \cap (X \ominus \check{B})] \oplus A]^c \\
&= [(X^c \ominus \check{A}) \cap (X \ominus \check{B})]^c \ominus A \\
&= [(X^c \ominus \check{A})^c \cup (X \ominus \check{B})^c] \ominus A \\
&= [(X \oplus A) \cup (X^c \oplus B)] \ominus A \, .
\end{aligned}$$

The "dual HMT" is the union of two dilations: of X by the hit SE (A), and of X^c by the miss SE (B). It consists of all points in an image X where either the translated hit SE $\check{A}$ intersects at least one ON pixel or the translated miss SE $\check{B}$ intersects at least one OFF pixel. Thus, the "dual HMT" is the set of pixels where there is at least a partial match to the SE, and the generalized foreground closing is a "dual HMT" followed by erosion by the hit SE. The generalized foreground closing, defined in (20), can also be visualized as the set complement of a generalized foreground opening on the background. In this view, it is the set complement of the union of hits for all matches to the set complement image (where the SE must also be spatially inverted).



(a) (b)

Figure 5. (a) Gen. foreground closing with SE of Figure 1a.
(b) Gen. background closing with SE of Figure 1a.

Figure 5a shows the action of Φ on the image in Figure 2, using the SE in Figure 1a as before. The lines of OFF pixels in Figure 5a are produced by first finding those points in the image where the line of hits intersects OFF pixels and the single miss intersects an ON pixel (this occurs on the top edges of the lines and on the top pixels of the text), and then eroding the result by the line of hits. Note that the resulting lines of connected OFF pixels are continuous in the text section. We next give several properties of the generalized foreground closing.

PROPERTY 9 . *The generalized foreground closing is extensive; i.e.,*

$$\Phi(X;A,B) \supseteq X \bullet A \supseteq X \ . \tag{21}$$

Proof. Since $(X \oplus A) \cup (X^c \oplus B) \supseteq X \oplus A$, and since erosion is a monotone increasing operator (i.e., $X \subseteq Y \Longrightarrow X \ominus A \subseteq Y \ominus A$), we have

$$\Phi(X) \supseteq (X \oplus A) \ominus A = X \bullet A \supseteq X.$$

PROPERTY 10 . *The generalized foreground closing by (A,B) is a fixed point of the closing by A; i.e.,*

$$\Phi(X;A,B) \bullet A = \Phi(X;A,B) \tag{22}$$

Proof. Since $(S \ominus A) \bullet A = S \ominus A$ for any set S, the above property immediately follows.

PROPERTY 11 . *The generalized foreground closing is idempotent; i.e.,*

$$\Phi[\Phi(X)] = \Phi(X) \ . \tag{23}$$

Proof. This follows from (19) and the idempotence of the generalized foreground opening:

$$\begin{aligned}\Phi[\Phi(X;A,B)] &= [\Psi([\Phi(X;A,B)]^c;\check{A},\check{B})]^c \\ &= [\Psi([\Psi(X^c;\check{A},\check{B})];\check{A},\check{B})]^c \\ &= [\Psi(X^c;\check{A},\check{B})]^c \\ &= \Phi(X;A,B)\ .\end{aligned}$$

As for the generalized openings, we define the *generalized background closing* of X by (A,B), denoted by $\phi(X;A,B)$ or simply by $\phi(X)$ if (A,B) are implied, as follows:

$$\phi(X;A,B) = [\psi(X^c;\check{A},\check{B})]^c \tag{24}$$

By working in a similar way as for the foreground closing it can be shown that the generalized background closing is a "dual HMT" followed by an erosion by the miss SE; i.e,

$$\phi(X;A,B) = \underbrace{[(X\oplus A)\cup(X^c\oplus B)]}_{\text{dual HMT}}\ominus B \tag{25}$$

Figure 5b shows the action of the generalized background closing on the image in Figure 2, using the SE in Figure 1a. The background in Figure 2 is contained within this set.

PROPERTY 12 . *The generalized background closing of X contains X^c; i.e.,*

$$\phi(X;A,B) \supseteq X^c\bullet B \supseteq X^c\ . \tag{26}$$

Proof. Since $(X\oplus A)\cup(X^c\oplus B)\supseteq X^c\oplus B$, and since erosion is increasing, we have

$$\phi(X)\supseteq(X^c\oplus B)\ominus B = X^c\bullet B\supseteq X^c.$$

PROPERTY 13 . *The generalized background closing by (A,B) is a fixed point of the closing by B; i.e.,*

$$\phi(X;A,B)\bullet B = \phi(X;A,B) \tag{27}$$

Proof. Same as the proof of (22).

PROPERTY 14 . *The generalized background closing of X by (A,B) is equal to the generalized foreground closing of X^c by (B,A); i.e.,*

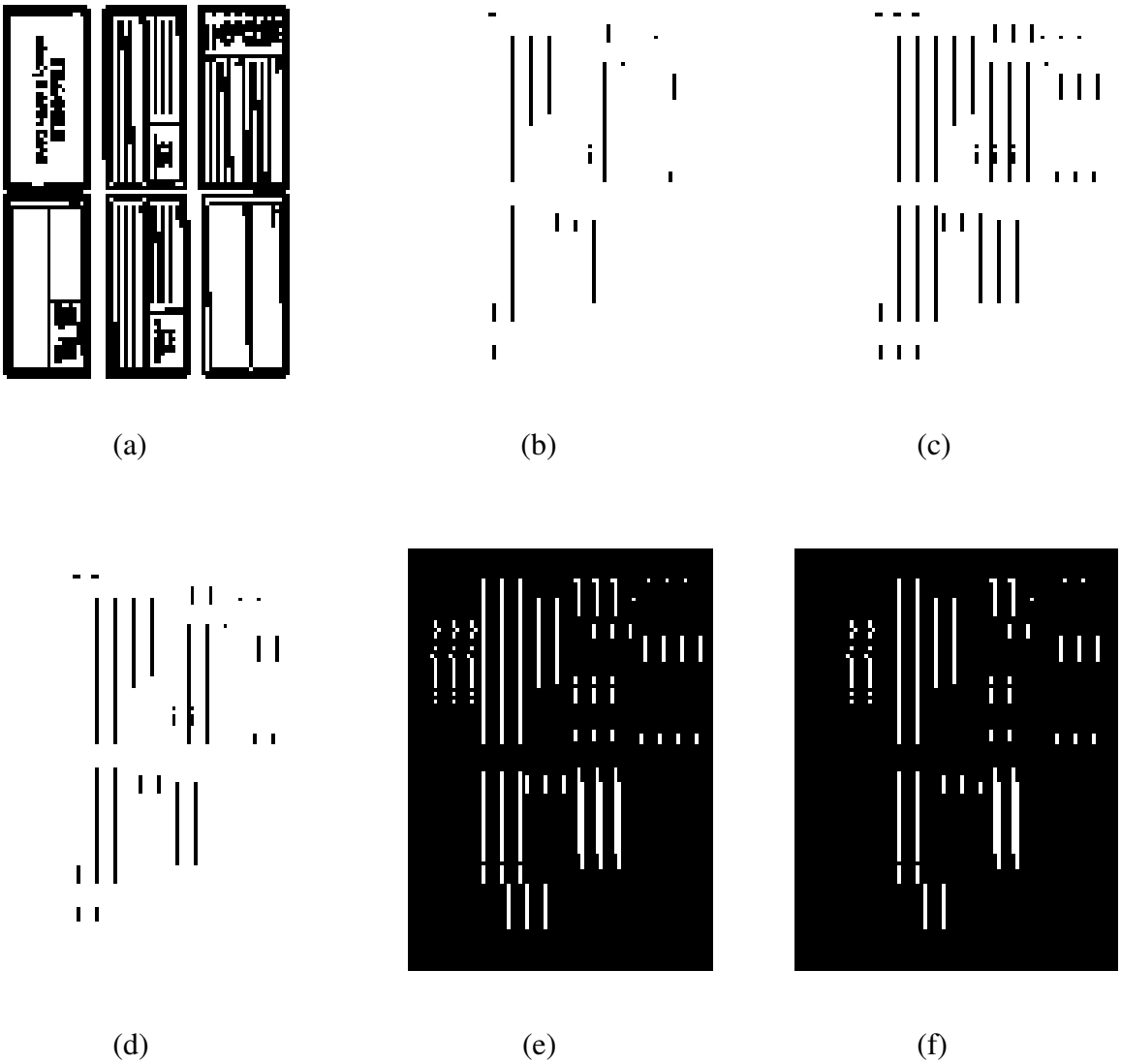
$$\phi(X;A,B) = \Phi(X^c;B,A)\ . \tag{28}$$

Proof. This follows from the definitions (19) and (24).

So far we have seen that the two generalized openings Ψ and ψ yield two sets whose union is a subset of the original information; i.e., $\Psi(X) \subseteq X$ and $\psi(X) \subseteq X^c$. By contrast, the two generalized closings Φ and ϕ yield two sets whose union is larger than the original information; i.e., $\Phi(X) \supseteq X$ and $\phi(X) \supseteq X^c$.

We next show the ability of the generalized openings and closings to extract textural patterns. Consider the hit-miss SE, given in Figure 6, which has a short-range periodic texture. Figure 7 shows the application of the HMT and the generalized openings and closings, using this SE, to an image. The strength of the HMT signal (b) should be contrasted with that of the generalized foreground opening in (c), which shows all the foreground pixels in (a) that participate in the HMT match. The generalized background opening in (d) shows background pixels in (a) that participate in the HMT match. The OFF pixels in the generalized foreground closing (e) are background pixels in (a) that are not selected by the dual HMT followed by an erosion of thel hit part of the SE. In this example, the number of such pixels is large and comparable to that of the foreground pixels in (c). Finally, the OFF pixels in the generalized background closing (f) are foreground pixels in (a) that are not selected by the dual HMT followed by an erosion of the miss part of the SE. From this example, it is apparent that these operations can extract signals corresponding to textural patterns within images that are of greater strength than may be expected from visual observation.

Figure 6. Hit-miss SE for short period texture.



(a) (b) (c)

(d) (e) (f)

Figure 7. (a) Starting image. The thin vertical lines are one pixel wide. Resolution: 48 pixels/inch. (b) HMT. (c) Generalized foreground opening. (d) Generalized background opening. (e) Generalized foreground closing. (f) Generalized background closing.

4 HMT with Rank Order Filters

The HMT of X by (A, B) detects the set of pixel locations at which A occurs in X and B occurs in X^c. Thus the HMT is a binary matched filter that acts simultaneously both on the image foreground and its background. Because it attempts to perform an exact matching, it is sensitive to noise, occlusions of image parts, or uncertainties about the exact shape of the hit or miss SE. To make it more robust one could perform a union of HMTs, each with slightly different SEs. This is the shape recognition approach followed in [1]; however, it could be computationally very intense due to the potentially large number of SEs.

Blur Matching: A more efficient method for improving the robustness of the match is to compute a *blur* match. In distinction with the exact match of an HMT, we define a blur-match HMT *(BHMT)* to require that (1) there is an ON pixel within a radius r_1 of each hit, and (2) there is an OFF pixel within a radius r_2 of each miss. For SEs that describe shapes to be matched in the image, the blur match gives immunity to pixel noise that occurs near the shape boundaries. Fortunately, this blur match is computed efficiently by first dilating the image by a disk SE R_1 of radius r_1 and dilating the set complement image by a disk SE R_2 of radius r_2, before computing the intersection of erosions in (3) for the HMT:

$$BHMT(X; A, B; R_1, R_2) = [(X \oplus R_1) \ominus A] \cap [(X^c \oplus R_2) \ominus B] \tag{29}$$

A *generalized foreground blur opening* is then generated by following the BHMT by a dilation by A:

$$\Psi_{blur}(X; A, B; R_1, R_2) = [((X \oplus R_1) \ominus A) \cap ((X^c \oplus R_2) \ominus B)] \oplus A \ . \tag{30}$$

PROPERTY 15 . *The generalized foreground blur opening by (A, B), with blurring disk SEs R_1 and R_2, is a fixed point of the opening by A; i.e.,*

$$[\Psi_{blur}(X; A, B; R_1, R_2)] \circ A = \Psi_{blur}(X; A, B; R_1, R_2) \ . \tag{31}$$

Proof. Same as the proof for (6).

A more flexible approach for constructing a robust generalized opening is to replace the erosions in the HMT with more general filters. There are two such generalizations, which, although different in their definition and implementation, are theoretically equivalent. These are the *threshold convolution* and the *rank order filtering* approach. In the remainder of this section, we will assume that we deal only with *discrete* signals.

Threshold Convolution: Let us represent the set X with a 2-D binary signal $x(n)$, where n is a pixel vector; i.e., $x(n) = 1$ if $n \in X$ and $X(n) = 0$ if $n \in X^c$. Similarly, let $w(n)$ be the the binary signal representing a finite SE W, which is also viewed as a window of pixels. Consider the following thesholded convolution of the input signal $x(n)$, which yields a binary-to-binary signal operation:

$$H[x * w(n) - \theta] \tag{32}$$

where $*$ denotes convolution, H is Heaviside's unit step function equal to $H(s) = 1$ if $s \geq 0$ and $H(s) = 0$ if $s < 0$, and θ is a variable threshold. If $\theta = |W|$ where $|W|$ denotes the number of pixels in W, then (32) is equivalent to the erosion of X by W. However, if $\theta < |W|$, then the threshold convolution imposes looser conditions than erosion on detecting W in X, and hence it could potentially be more robust by adjusting θ. These ideas have been applied to several template matching approaches for binary object detection, as described in [3, 10, 8].

Rank Order Filtering: Given a discrete-time signal $f(n)$ and a finite window W, the r-th rank order transformation of f by W yields the signal

$$f \square_r W(n) = r - \text{th largest of } f(n+k), \quad k \in W \tag{33}$$

where $r = 1, 2, ..., |W|$. Applying rank order filters to the binary signal $x(n)$ representing X yields a binary signal too.[1] In [6] a set-theoretic definition of binary rank order filters was given that avoids sorting and uses only pixel counting; thus the r-th rank order transformation of X by W is

$$X \square_r W = \{z : |X \cap (W + z)| \geq r\} \tag{34}$$

Note that if $r = |W|$, then $X \square_r W$ becomes the erosion $X \ominus W$; for $r = 1$ we get a dilation. Obviously, the threshold convolution (32) and the binary rank order filtering (34) yield identical signals if $\theta = r$.

In [5] it was shown that the convolution of a binary signal f with a binary template w (representing a SE W) compared to a threshold, or its equivalent rank order operation, is the optimum (using a Bayesian formulation) decision for detecting w in f, when f contains a shifted version of w corrupted with binary salt-and-pepper noise. In [9] a composition of rank order filter and a dilation was proposed for feature detection as a robust replacement of the ordinary opening. This r-th *rank-dilation* operation of a (binary or non-binary) signal f by a binary template (set) W consists of the r-th rank order transformation of f by W followed by a dilation by W. Thus the rank order operation replaces the erosion in an ordinary opening and (by varying $r = 1, ..., |W|$) serves in detecting W more robustly than the erosion, whereas the dilation redraws W at the detected locations. The same operation was called *rank opening* in [5] and *shape inference opening* in [12]. In [9] the rank-dilation operation was further superimposed with the original image using pointwise minimum, which makes it anti-extensive and idempotent.[2]

Motivated by the above ideas, where rank order operations can improve the performance of matched filters when they replace erosions, we will use rank order filters in the HMT of the generalized opening. Thus, for a given disjoint SE pair (A, B), we define the (p, q)-th *rank hit-miss transform* of X by (A, B) as the set transformation

$$X \otimes_{p,q} (A, B) = (X \square_p A) \cap (X^c \square_q B) \tag{35}$$

[1] Binary rank order filters were used in [2]. Rank order filters for non-binary signals were used in [4, 7]. For relationships between rank order and morphological operations see [6].

[2] In general, note that if $\Gamma(\cdot)$ is any increasing set operation, then the operation $X \rightarrow X \cap \Gamma(X)$ is increasing, anti-extensive, and idempotent.

where $p = 1, 2, ..., |A|$ and $q = 1, 2, ..., |B|$. Then, the (p, q)-th *generalized foreground rank opening* of X by (A, B) is the set transformation

$$\Psi_{p,q}(X; A, B) = [(X \square_p A) \cap (X^c \square_q B)] \oplus A \tag{36}$$

Thus $\Psi_{p,q}$ consists of a rank HMT followed by dilation with the hit SE. Similarly, we can define the rank order versions of the other generalized openings and closings.

PROPERTY 16 . *The generalized foreground rank opening by (A, B) is a fixed point of the opening by A; i.e.,*

$$[\Psi_{p,q}(X; A, B)] \circ A = \Psi_{p,q}(X; A, B) \ . \tag{37}$$

Proof. Same as the proof for (6).

PROPERTY 17 . *The generalized foreground rank opening by (A, B) is not in general idempotent, except in the case where $p = |A|$ and $q = |B|$, where it reduces to the previously defined generalized foreground opening Ψ.*

Proof. Consider an image with m ON pixels in a horizontal line, and a SE A with $n < m$ horizontal hits. If $p < n$, the rank order operation will cause the image to shrink to $m - p + 1$ pixels, and the subsequent dilation expands it to $m + n - p$ pixels, which is larger than m pixels. Each subsequent application of the generalized rank order opening will also increase the image by $n - p$ pixels.

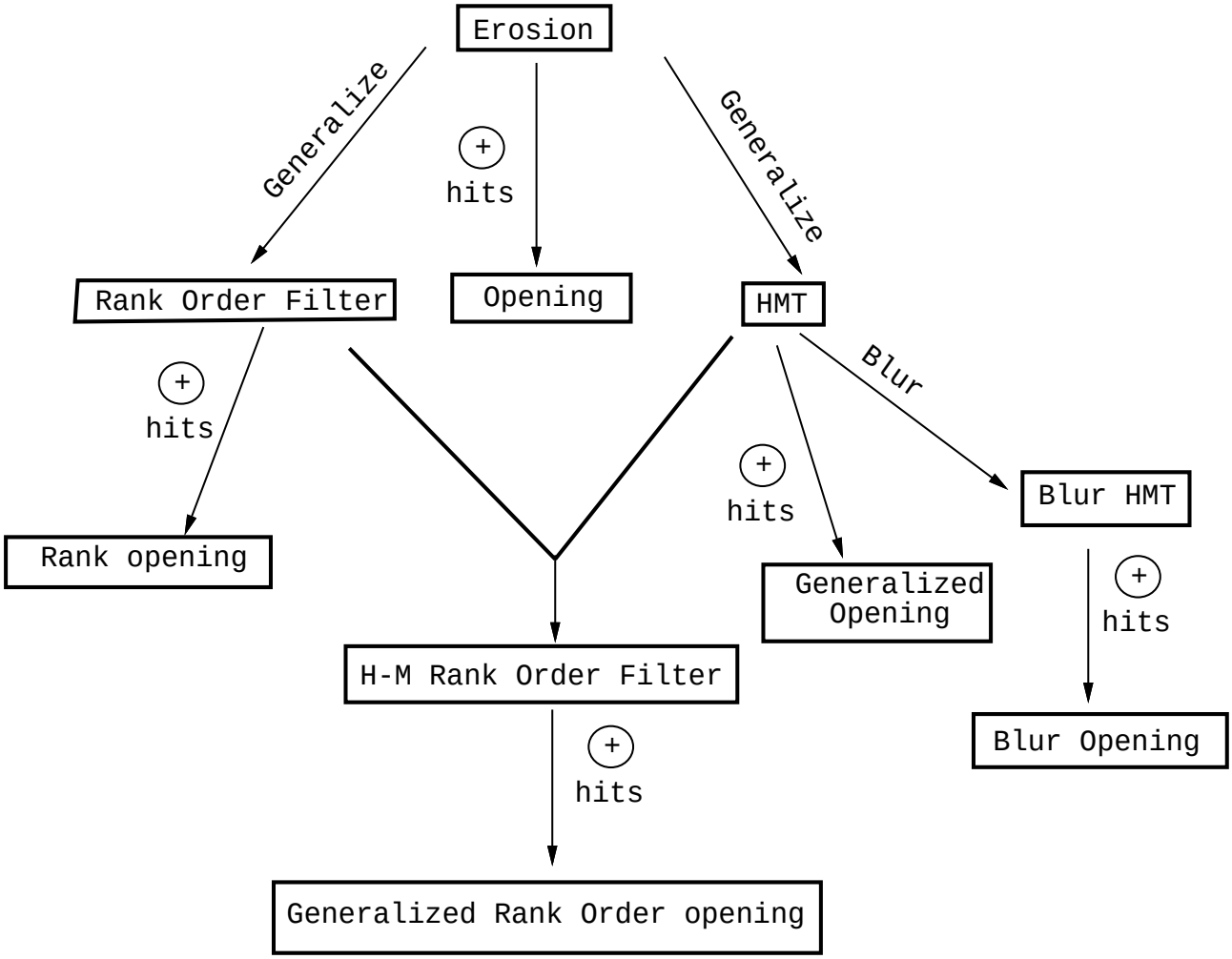


Figure 8. Generalizations of the Opening, HMT, and Rank Order Filter.

These various extensions are summarized in Figure 8. The erosion can be generalized in three ways: by forming an opening, by choosing a rank order smaller than the cardinality of the SE, and by forming a HMT. Both the rank order filter and the HMT can be generalized to an opening by dilating with the hits in the SE. The HMT can be generalized to a blur HMT, which can be further generalized to a blur opening. And the rank order filter and HMT can be combined to form a HM rank order filter, which can be further generalized to an opening by dilating with the hits. Finally,

not shown in Figure 8, a rank order operation can be applied to a blur HMT, which can then be opened to form a generalized rank order blur opening.

These operations are of more than academic interest. The all-or-nothing character of the erosion and HMT complicate their use with highly variable image data. However, the blur and rank order filters, used separately or in conjunction, provide sufficient flexibility to permit reliable statistics to be accumulated on noisy or variable data.

Finally, note that all the ideas presented in this section on rank HMTs and generalized rank openings of a (binary image) set X can be extended easily to gray-level images f by replacing all rank order set transformations with rank order filters acting on gray-level images; replacing $\cap$ with pointwise minimum; and replacing X^c with $m - f$ where m is the maximum gray amplitude.

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